Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-04-01

Conceptual Review

Question 1. Let $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Rewrite $\mathrm{d}V$ in terms of...

(a) z, r, θ.
(b) ρ, ϕ, θ.
(c) u, v, w where $x = u + vw$, $y = v + wu$, and $z = w + uv$.

Question 2. The transformation $x = -2u$, $y = 2v + u^2$ maps a region S of the uv-plane onto a region R of the xy-plane. If R has area 10, what is the area of S?

Problems

Problem 1. Use spherical coordinates to compute the volume inside of the unit sphere (a sphere of radius 1).

Problem 2. Find the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Hint: Use the transformation $x = au$, $y = bv$, $z = cw$, and then the preceding problem.

Problem 3. In this problem, we will consider the integral

$$(*) \qquad \int_0^1 \int_0^1 \frac{1}{1 - xy}\,\mathrm{d}x\,\mathrm{d}y.$$

This is an interesting integral for the following reason: we can expand

$$\frac{1}{1 - xy} = 1 + xy + x^2y^2 + x^3y^3 + \cdots$$

as a geometric series, and as you can easily verify,

$$\int_0^1 \int_0^1 x^n y^n\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{(n+1)^2}.$$

So the value of the integral should be equal to the value of the very famous infinite series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

whose exact value was first determined by Leonhard Euler.

(a) Try to evaluate the integral (*) directly. You should find that the inner integral is doable, but that the outer integral is problematic.
(b) Use the change of variables $x = u + v$ and $y = u - v$ to rewrite the integral with integration order $\mathrm{d}v\,\mathrm{d}u$. You will need two integrals (once you sketch the relevant region in the uv-plane you will see why).
(c) ** Evaluate the integrals if you are feeling ambitious, to discover Euler's famous result (although this is not how he proved it).

Here are some brief answers or comments on the exercises. As always, I am willing to elaborate further on request.

Question 1.

(a) $r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$
(b) $\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$
(c) $|1-u^2-v^2-w^2+2uvw|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$

Question 2. In this case, one finds the Jacobian determinant to be the constant 4. So the answer is $10/4 = 5/2$.

Note that this problem would not be solvable as written if the Jacobian determinant had been a function depending on u and v instead. (Then the answer would depend on the specific region R, and not only on the area of R.)

Problem 1. $4\pi/3$

Problem 2. In the uvw coordinate system, the corresponding region is that enclosed in the unit sphere, which we saw from the preceding problem to be $4\pi/3$. The absolute value of the Jacobian determinant is just $|abc|$, which is a constant. Thus the volume is $|abc|4\pi/3$.

Problem 3.

(a) Omitted

(b) $\displaystyle\int_0^{1/2}\int_{-u}^{u}\frac{2\,\mathrm{d}v\,\mathrm{d}u}{1-u^2+v^2}+\int_{1/2}^{1}\int_{u-1}^{1-u}\frac{2\,\mathrm{d}v\,\mathrm{d}u}{1-u^2+v^2}$. Here the absolute value of the Jacobian determinant is 2, and the original region was defined by $0 \le x \le 1$ and $0 \le y \le 1$, which becomes $0 \le u+v \le 1$ and $0 \le u-v \le 1$.

(c) The computation is shown below. We use the antiderivative

$$\int \frac{\mathrm{d}t}{a^2+t^2} = \frac{1}{a}\arctan(t/a) + C.$$

$$\begin{aligned}
&\int_0^{1/2}\int_{-u}^{u}\frac{2\,\mathrm{d}v\,\mathrm{d}u}{1-u^2+v^2}+\int_{1/2}^{1}\int_{u-1}^{1-u}\frac{2\,\mathrm{d}v\,\mathrm{d}u}{1-u^2+v^2}\\
&= 2\left[\int_0^{1/2}\frac{\arctan v/\sqrt{1-u^2}}{\sqrt{1-u^2}}\Bigg|_{v=-u}^{v=u}\mathrm{d}u+\int_{1/2}^{1}\frac{\arctan v/\sqrt{1-u^2}}{\sqrt{1-u^2}}\Bigg|_{v=u-1}^{v=1-u}\mathrm{d}u\right] \qquad \text{by using } a=\sqrt{1-u^2} \text{ in the above}\\
&= 4\left[\int_0^{1/2}\frac{\arctan u/\sqrt{1-u^2}}{\sqrt{1-u^2}}\,\mathrm{d}u+\int_{1/2}^{1}\frac{\arctan\,(1-u)/\sqrt{1-u^2}}{\sqrt{1-u^2}}\,\mathrm{d}u\right] \qquad \text{because arctan is an odd function}\\
&= 4\left[\int_0^{\pi/6}\frac{\arctan(\sin(t)/\sqrt{1-\sin(t)^2})}{\sqrt{1-\sin(t)^2}}\cos(t)\,\mathrm{d}t+\int_{\pi/3}^{0}\frac{\arctan((1-\cos(t))/\sqrt{1-\cos(t)^2})}{\sqrt{1-\cos(t)^2}}(-\sin(t))\,\mathrm{d}t\right]\\
&= 4\left[\int_0^{\pi/6} t\,\mathrm{d}t+\int_{\pi/3}^{0}-\arctan\left(\frac{1-\cos(t)}{\sin(t)}\right)\mathrm{d}t\right]\\
&= 4\left[\frac{\pi^2}{72}-\int_{\pi/3}^{0}\arctan(\tan(t/2))\,\mathrm{d}t\right] = 4\left[\frac{\pi^2}{72}-\left(-\frac{\pi^2}{36}\right)\right] = \boxed{\pi^2/6}.
\end{aligned}$$